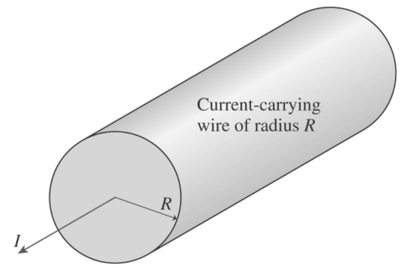
PHYS 2212
Recitation 09 March 29 – April 1

Example

An infinitely long current-carrying cable has a radius R and a current density that varies with distance r from the central axis according to

$$\vec{J} = J_0 \frac{R}{r} \cos\left(\frac{\pi r}{R}\right) \hat{z}$$



where J_0 is a constant. Find an expression for the magnitude of the magnetic field as a function of r inside the wire.

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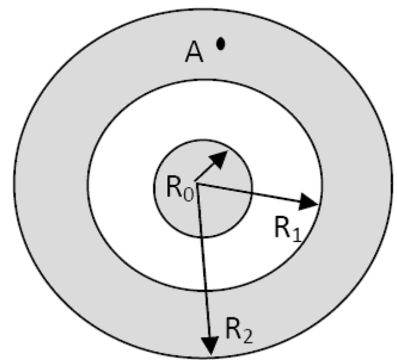
Problem

A coaxial cable consists of a solid inner wire of radius R_0 and a hollow outer wire with inner radius R_1 and outer radius R_2, separated by a hollow gap as shown. The inner wire carries a current I_0 out of the page in the positive z direction. The outer wire has a current density that varies with distance r' from the central axis according to

$$\vec{J} = -J_0 \frac{R}{r'} \hat{z}$$



Calculate the magnetic field within the outer cable at point A (indicted), which is a distance $R_1 < r < R_2$ from the central axis.

PHYS 2212 **Solution**
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Example

An infinitely long current-carrying cable has a radius R and a current density that varies with distance r from the central axis according to

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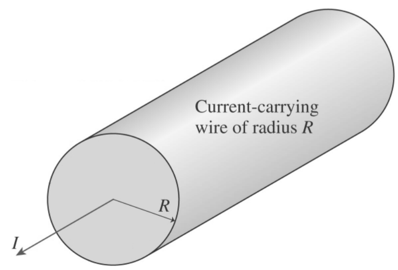
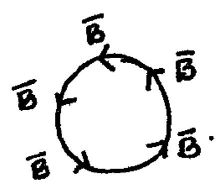


where J_0 is a constant. Find an expression for the magnitude of the magnetic field as a function of r inside the wire.

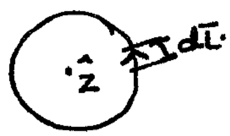
We use Ampere's Law to calculate the magnetic field inside the current cable, say at a distance r from the center.

Ampere's Law states that the "circulation of magnetic field along a closed loop is equal to μ_0 times the total current crossing the loop". $\Rightarrow$ $\oint_{\text{Loop}} \bar{B} \cdot dL = \mu_0 I$



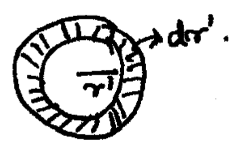
By arguments of symmetry we conclude that the field is along the tangential direction everywhere, equal in magnitude at a given "r" for all ϕ i.e there would be no angular dependence on the magnitude.

I assume the direction out of the page to be the positive $\hat{z}$, thus the loop with this direction is "a counterclockwise" one.



$$\therefore \quad \bar{B}(r)\cdot dL = 2\pi r B$$

Now the current enclose inside this loop of radius "r" is $\displaystyle\int_{r'=0}^{r} \bar{J}(r')\cdot d\bar{A}$ $\qquad d\bar{A} = 2\pi r'\,dr'\ \hat{z}$ (see that it is a vector)



$\odot +\hat{z}$.

$$\therefore \quad I(r) = \int_0^r J_0 \frac{R}{\underline{r'}} \cdot \cos\frac{\pi r'}{R}\, dr' \cdot 2\pi \underline{r'} \quad \hat{z}\cdot\hat{z}$$

$$= J_0 R\, 2\pi \int_0^r \cos\frac{\pi r'}{R}\, dr' = 2\pi J_0 R \left.\frac{\sin\frac{\pi r'}{R}}{\pi/R}\right|_0^r$$

$$= 2 J_0 R^2 \sin\frac{\pi r'}{R}\Big|_0^r = 2 J_0 R^2 \sin\frac{\pi r}{R}. = I(r)$$

$$\therefore \quad B(r<R)\, \cancel{2}\pi r = \left\{ \cancel{2} J_0 R^2 \sin\frac{\pi r}{R} \right\} \mu_0$$

$$\therefore \quad B(r<R) = \frac{\mu_0 J_0 R^2 \sin\frac{\pi r}{R}}{\pi r}$$

$B>0 \Rightarrow$ in the positive direction of loop $\Rightarrow$ +ve $\hat{\phi}$ direction.

Problem

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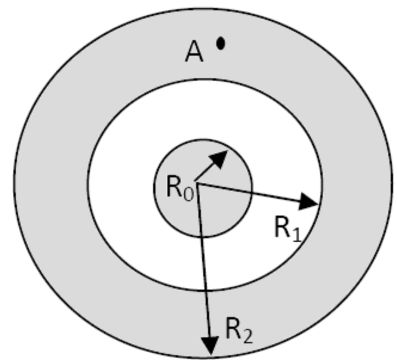
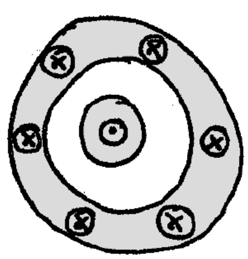


Calculate the magnetic field within the outer cable at point A (indicted), which is a distance $R_1 < r < R_2$ from the central axis.

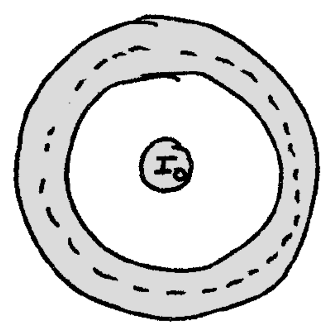
Firstly, we need to realise that the currents in the "core" cable (R_0) and the outer cable ($R_1 < r < R_2$) are in opposite directions.



$\otimes$ ≡ into the page ($-\hat{z}$)

$\odot$ ≡ out of the page ($+\hat{z}$).

We take a loop in the "counter clockwise" sense. The positive current density direction would then be $+z$.



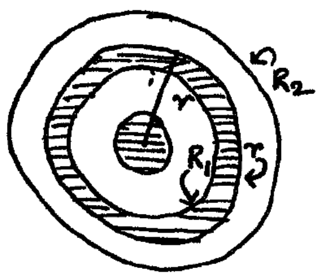
By Arguments of symmetry we again realise that the magnetic field has only a $\hat{\phi}$ direction and a "radial" dependence i.e $\bar{B} = B(r)\,\hat{\phi}$.

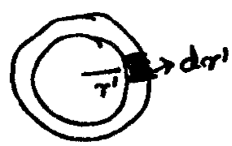
Construct an Amperian loop at a distance "r" within the outer cable $\Rightarrow r \in [R_1, R_2]$.

From Ampere's Law we know that

$$\oint_{loop} \bar{B}\cdot d\bar{l} = \mu_0 I_{enc} \quad \text{and} \quad \oint \bar{B}\cdot d\bar{l} = B(r)\, 2\pi r$$

The I_{enc} has 2 "contributions". One is due to the current through inner cable and the other is through the outer cable within $R_1 < r$





$dA = 2\pi r' dr'$

$$\therefore \quad I_{enc} = \underbrace{+I_0}_{\text{since along } +z} + \int_{R_1}^{r} \bar{J}\cdot d\bar{A} \quad \Big| \; d\bar{A} = 2\pi r' dr' \underbrace{\hat{z}}_{(+ve\ \hat{z})}$$

$$= I_0 + \int_{R_1}^{r} J_0 \frac{R}{r'} \cdot 2\pi r' \, dr' \, (-\hat{z}\cdot\hat{z})$$

$$= I_0 - J_0 R\,(2\pi) \int_{R_1}^{r} dr' = I_0 - J_0 2\pi R\,(r - R_1).$$

$\therefore \quad B(r)\, 2\pi r = \{ I_0 - J_0 2\pi R (r - R_1) \} \mu_0$ #

$$B(r) = \frac{I_0 \mu_0}{2\pi r} - \frac{\mu_0 J_0 R (r - R_1)}{r}$$

both the forms of the answer have some very vital information.

ⓐ $B(r)$ could be positive or negative. If it is positive, it means that $\bar{B}$ is along the positive sense of the loop (counterclockwise). If $B(r)$ is negative $\Rightarrow -|B(r)|\hat{\phi}$, as we can see it is along clockwise direction.

ⓑ You can observe the "contribution" to $\bar{B}$ comes from the inner wire carrying a current I_0 i.e $\frac{\mu_0 I_0}{2\pi r}$ as well as from the current in the cable between "R_1 & r".

ⓒ See that R_2 does not figure in the answer, showing that what matters is the "current enclosed" only.